

WIENER-WINTNER FOR HILBERT TRANSFORM

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ABSTRACT. We prove the following extension of the Wiener–Wintner Theorem and the Carleson Theorem on pointwise convergence of Fourier series: For all measure preserving flows (X, μ, T_t) and $f \in L^p(X, \mu)$, there is a set $X_f \subset X$ of probability one, so that for all $x \in X_f$ we have

$$\lim_{s \downarrow 0} \int_{s < |t| < 1/s} e^{i\theta t} f(T_t x) \frac{dt}{t} \quad \text{exists for all } \theta.$$

The proof is by way of establishing an appropriate oscillation inequality which is itself an extension of Carleson’s theorem.

1. THE MAIN THEOREM

We are concerned with quantitative inequalities related to the pointwise convergence of singular integrals that are uniform with respect to modulation. To state our results, define dilation and modulation operators by

$$(1.1) \quad \text{Dil}_s^{(p)} f(x) \stackrel{\text{def}}{=} s^{-1/p} f(x/s), \quad 0 < s, p < \infty.$$

$$\text{Dil}_s^{(\infty)} f(x) \stackrel{\text{def}}{=} f(x/s), \quad 0 < s < \infty.$$

$$(1.2) \quad \text{Mod}_\xi f(x) \stackrel{\text{def}}{=} e^{ix\xi} f(x), \quad \xi \in \mathbb{R}.$$

Let K be a distribution. The most important example will be $K_H(y) \stackrel{\text{def}}{=} \zeta(y) \frac{1}{y}$, where ζ is a smooth, symmetric, compactly supported function. This is a distribution associated to a truncation of the Hilbert transform kernel.

Our principal concern is the convergence of terms $(\text{Dil}_s^{(1)} K) * f(x)$ in a pointwise sense, and in one that is, in addition, uniform over all modulations. To do this, we use the following definition.

$$(1.3) \quad \text{Osc}_n(K; f)^2 \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \sup_{k_j \leq l < l' < k_{j+1}} |[(\text{Dil}_{2^{l/n}}^{(1)} K) - (\text{Dil}_{2^{l'/n}}^{(1)} K)] * f|^2.$$

Research supported in part by the National Science Foundation. The author is a Guggenheim Fellow.

This definition depends upon a choice of an increasing sequence of integers $k_j \in \mathbb{Z}$, a dependence that we suppress as relevant constants are independent of the choice of $\{k_j\}$. It also depends upon a choice of positive integer n , which we have incorporated into the notation. This only permits dilations of the form $2^{l/n}$ for integers l .

1.4. Theorem. *Fix a smooth, symmetric, compactly supported function ζ . For integers $n > 0$ and $1 < p < \infty$ there is a constant $C_{n,p,\zeta}$ so that we have the inequality*

$$(1.5) \quad \left\| \sup_N \text{Osc}_n(K_H; \text{Mod}_N f) \right\|_p \leq C_{n,p,\zeta} \|f\|_p.$$

The inequality holds for all choices of increasing sequences $\{k_j : j \geq 1\}$ satisfying $k_{j+1} \geq k_j + n$.

Our primary interest in this theorem is the corollary below, which is a Hilbert transform counterpart to the well known Wiener–Wintner theorem for ergodic averages. Deriving the corollary below is a standard part of the literature, with the roots of the argument going back to Calderón [8]. The use of an oscillation inequality to establish convergence was introduced by Bourgain [7]. Also see the papers of Campbell et al. [10], and Jones et al. [14].

1.6. Corollary. *For all measure preserving flows $\{T_t : t \in \mathbb{R}\}$ on a probability space (X, μ) and functions $f \in L^p(\mu)$, there is a set $X_f \subset X$ of probability one, so that for all $x \in X_f$ we have*

$$\lim_{s \downarrow 0} \int_{s < |t| < 1/s} e^{i\theta t} f(T_t x) \frac{dt}{t} \quad \text{exists for all } \theta.$$

This is a common extension of two classical theorems: Carleson’s Theorem [11] on Fourier series with Hunt’s extension [13], and the Wiener–Wintner Theorem [22] on ergodic averages.

Carleson’s Theorem. *We have the inequality*

$$\left\| \sup_N \left| \int \text{Mod}_N f(x - y) \frac{dy}{y} \right| \right\|_p \lesssim \|f\|_p, \quad 1 < p < \infty.$$

Wiener–Wintner Theorem. *For all measure preserving flows $\{T_t : t \in \mathbb{R}\}$ on a probability space (X, μ) and functions $f \in L^p(X, \mu)$, there is a set $X_f \subset X$ of probability one, so that for all $x \in X_f$ we have*

$$\begin{aligned} \lim_{s \rightarrow \infty} s^{-1} \int_{-s}^s e^{i\theta t} f(T_t x) dt & \quad \text{exists for all } \theta \\ \lim_{s \rightarrow 0} s^{-1} \int_{-s}^s e^{i\theta t} f(T_t x) dt & \quad \text{exists for all } \theta. \end{aligned}$$

The Wiener–Wintner Theorem can be seen as an extension of the Birkhoff Ergodic Theorem. The Carleson Theorem is a deep result from the 60’s, and since then several proofs have been offered. An extensive survey and bibliography on this subject can be found in [15].

The possibility of extending the Wiener–Wintner Theorem to the setting of the Hilbert transform was first raised in the paper of Campbell and Petersen [9]. The specific result proved there was essentially Carleson’s Theorem on the integers, with a transference to measure preserving systems. Part of this was contained in a prior work of Máté [18], a work that was overlooked until much later.

Assani [1, 2] proved our Corollary 1.6 on different classes of dynamical systems. Indeed, he formulated the concept of a *Wiener–Wintner system*. In this nomenclature, our corollary states that all measure preserving systems are Wiener–Wintner systems.

Our tool to prove convergence in the Hilbert transform setting is the oscillation inequality (1.5), an idea first employed in ergodic theory in the pioneering work of Bourgain on the ergodic theorem along arithmetic sequences [7]. The use of oscillation has subsequently been systematically studied in e.g. [10, 14] and in references therein.

The main goal of the paper is a proof of Theorem 1.4. Clearly, we follow the lines of a proof of Carleson’s Theorem. In particular we employ the Lacey–Thiele approach [17] and refine one part of it to deduce our main theorem. We will also appeal to the ‘restricted weak type argument’ of C. Muscalu, T. Tao, and C. Thiele [20] and L. Grafakos, T. Tao, and E. Terwilleger [12].

Acknowledgment. The authors have benefited from conversations with Jim Campbell, Anthony Quas and Mate Wierdl. Part of this research was completed at the Schrödinger Institute, Vienna Austria. For one of us (ML), discussions with Karl Petersen about this question formed our introduction to Carleson’s theorem, for which we have been indebted to him ever since.

2. DEDUCTION OF THEOREM 1.4

There are two more technical estimates that we prove. Specifically, let ψ be some Schwartz function which satisfies

$$(2.1) \quad 0 \leq \widehat{\psi}(\xi) \leq C_0,$$

$$(2.2) \quad \widehat{\psi} \text{ is supported in } [-2, -\tfrac{1}{2}],$$

$$(2.3) \quad |\psi(y)| \leq C_1 \min(|y|^{-\nu}, |y|^\nu).$$

Here, ν will be a large constant whose exact value we need not specify. And we will not have complete freedom in precisely which Schwartz function ψ we can take here. It should arise in a particular way described in the proof of Proposition 2.4, and will be nonzero! The purpose of this section is to describe how a particular result for *any choice of non zero ψ* as above will lead to a proof of our main theorem.

Consider the distribution

$$\Psi \stackrel{\text{def}}{=} \sum_{v=1}^{\infty} \text{Dil}_{2^{-v}}^{(1)} \psi.$$

We will prove the following two propositions in the next section.

2.4. Proposition. *With the assumptions (2.2)—(2.3), the inequality (1.5) holds with $n = 1$ and the distribution K_H replaced by Ψ .*

2.5. Proposition. *We have the inequality*

$$(2.6) \quad \left\| \sup_N \left[\sum_{j=-\infty}^{\infty} |\{\text{Dil}_{2^j}^{(1)} \psi\} * \text{Mod}_N(f)|^2 \right]^{1/2} \right\|_p \lesssim \|f\|_p, \quad 1 < p < \infty.$$

Note that for fixed modulation, (2.6) is a Littlewood Paley inequality, making the inequality above a “Carlesonized Littlewood Paley” inequality. Inequalities like this have been proved by Prestini and Sjölin [21]. They also follow from the method of Lacey and Thiele.

Both propositions follow from our Proposition 3.9 of the next section, which is phrased in a language conducive to the methods of Lacey and Thiele [17]. These methods have been applied in a number of variants of Carleson’s theorem, see e.g. Pramanik and Terwilleger [19] and Grafakos, Tao and Terwilleger [12].

We turn to the deduction of Theorem 1.4. Observe that the two previous propositions immediately prove that when we consider dilations which are powers of $2^{1/n}$ we have

$$\left\| \sup_N \text{Osc}_n(\Psi; \text{Mod}_N f) \right\|_p \lesssim n \|f\|_p, \quad n \in \mathbb{N}, \quad 1 < p < \infty.$$

Thus we need not concern ourselves with this feature of Theorem 1.4 and Corollary 1.6.

For a distribution K , set

$$\|K\|_{*,p} = \sup_{\|f\|_p=1} \left\| \sup_N \text{Osc}_1(K; \text{Mod}_N f) \right\|_p.$$

Note that since our definition incorporates differences, this is a seminorm on distributions K . That is, it obeys the triangle inequality (which we use), but can be zero for non zero distributions. In particular, for a Dirac point mass δ we have $\|\delta\|_{*,p} = 0$, and similarly for the distribution K with $\hat{K} = \mathbf{1}_{[0,\infty)}$.

Our task is to show that $\|K_H\|_{*,p} < \infty$, where $K_H(y) = y^{-1}\zeta(y)$ for some smooth symmetric, compactly supported Schwartz function. Our Proposition 2.4 is, with this notation, the assertion that $\|\Psi\|_{*,p} < \infty$. The same inequality will hold for a kernel which can be obtained as a convex combination of dilations of ψ and Ψ . Thus, set

$$\Psi_0 \stackrel{\text{def}}{=} \int_0^1 \text{Dil}_{2^s}^{(1)} \Psi \frac{ds}{s}.$$

In this integral, we are careful to integrate against the measure $\frac{ds}{s}$, which is the Haar measure for the positive reals under multiplication, the underlying group for the dilation operators. In particular, it follows that Ψ_0 is a distribution whose Fourier transform is a nonzero constant on $(-\infty, -1)$ and is 0 on $(-\frac{1}{2}, \infty)$. Thus by Proposition 2.4, we clearly have $\|\Psi_0\|_{*,p} < \infty$.

Now we will show that $\|D_0\|_{*,p} < \infty$ for the distribution

$$D_0(y) = y^{-1}\zeta(y) - c(\Psi_0(y) - \overline{\Psi_0(y)}),$$

where we choose the complex constant c so that $\lim_{\xi \rightarrow \infty} \widehat{D}_0(\xi) = 0$. In fact, it is a well known elementary fact that for $c = i\pi$,

$$(2.7) \quad \int \zeta(y) e^{i\xi y} \frac{dy}{y} = c + O(|\xi|^{-1}).$$

We will decompose the distribution D_0 into a sum which can be treated with Proposition 2.5. Then using that $\|\Psi_0\|_{*,p} < \infty$ and $\|D_0\|_{*,p} < \infty$, we obtain the desired inequality for $K(y) = y^{-1}\zeta(y)$.

Choose χ to be a smooth function supported on $\frac{1}{2} \leq |\xi| \leq 2$ so that

$$\sum_{k=-\infty}^{\infty} \text{Dil}_{2^{-k}}^{\infty} \chi = \mathbf{1}_{\mathbb{R}-\{0\}},$$

and set $\widehat{\Delta}_k = \widehat{D}_0 \text{Dil}_{2^{-k}}^{\infty} \chi$. The following lemma finishes the proof of Theorem 1.4.

2.8. Lemma. *We have*

$$\|\Delta_k\|_{*,p} \lesssim 2^{-|k|}, \quad k \in \mathbb{Z}.$$

Proof. We will verify that

$$(2.9) \quad \|\widehat{\Delta}_k\|_{\infty} \lesssim 2^{-|k|}, \quad k \in \mathbb{Z},$$

$$(2.10) \quad \widehat{\Delta}_k \text{ is supported on } 2^{-k-1} \leq |\xi| \leq 2^{-k+1},$$

$$(2.11) \quad |\Delta_k(y)| \lesssim 2^{-k-|k|} (1 + 2^{-k}|y|)^{-\nu}, \quad k \in \mathbb{Z}, y \in \mathbb{R},$$

with implied constants independent of $k \in \mathbb{Z}$ and ν the large, unspecified constant that appears in (2.3). With decay in $|k|$ in both (2.9) and (2.11), the lemma then follows from a trivial change of scale and from Proposition 2.5.

Let us recall the trivial estimate which follows from the symmetry of ζ ,

$$(2.12) \quad |\widehat{K_H}(\xi)| = \left| \int \zeta(y) \frac{e^{i\xi y}}{y} dy \right| \lesssim |\xi|.$$

In addition we have the estimate below, applied for $|\xi| \leq 1$

$$(2.13) \quad \left| \frac{d^w}{d\xi^w} \widehat{K_H}(\xi) \right| = \left| \int \zeta(y) y^{w-1} e^{i\xi y} dy \right| \lesssim \begin{cases} |\xi| & w \text{ even} \\ 1 & w \text{ odd.} \end{cases}$$

Whereas for $|\xi| \geq 1$, we have

$$(2.14) \quad \left| \frac{d^w}{d\xi^w} \widehat{K_H}(\xi) \right| \lesssim |\xi|^{-\nu}, \quad |\xi| > 1, \quad 0 < w \leq \nu.$$

That is, we have very rapid decay in a large number of derivatives.

Now, (2.10) is true by definition of Δ_k . To see (2.9) for $k \geq 2$, note that this is only determined by the Fourier transform of K_H since $\widehat{\Psi}_0$ and $\widehat{\bar{\Psi}}_0$ are zero. The result easily follows by the inequality (2.12) and property (2.10). For $k \leq 2$, the inequality follows from the construction of D_0 , and in particular the property in (2.7).

We turn to the last condition, (2.11). It is well known that decay of order ν in spatial variables is implied by differentiability of a function in frequency variables. Observe that

$$\begin{aligned} (y^\nu \Delta_k(y))^\wedge(\xi) &= i^{-\nu} \frac{d^\nu}{d\xi^\nu} \widehat{\Delta_k}(\xi) \\ &= i^{-\nu} \frac{d^\nu}{d\xi^\nu} \text{Dil}_{2^{-k}}^{(\infty)} \chi(\xi) \widehat{K_H}(\xi). \end{aligned}$$

Hence,

$$|(y^\nu \Delta_k(y))^\wedge(\xi)| \leq \sum_{w=0}^{\nu} 2^{kw} \left| \frac{d^{\nu-w}}{d\xi^{\nu-w}} \sup_{2^{-k-1} \leq |\xi| \leq 2^{-k+1}} \widehat{K_H}(\xi) \right|.$$

For $k \geq 1$, this sum is dominated by the last two terms. To control them, use (2.13), supplying the estimate $\lesssim 2^{(\nu-1)k}$. This is better by a factor of 2^{-k} than the trivial estimate, so that Fourier inversion proves (2.11) in this case.

The case of $k \leq 0$ is easier, due to the rapid decay in (2.14). □

3. DECOMPOSITION AND MAIN PROPOSITION

We state the definitions needed for the main proposition and conclude this section with the argument of how this proposition proves the results of the previous section, namely Proposition 2.4 and Proposition 2.5.

In addition to the modulation and dilation operators in (1.1) and (1.2), we need translation operators

$$(3.1) \quad \text{Tran}_y f(x) \stackrel{\text{def}}{=} f(x - y), \quad y \in \mathbb{R}.$$

We set \mathcal{D} to be the dyadic grid and say that $I \times \omega \in \mathcal{D} \times \mathcal{D}$ is a *tile* iff $|\omega| \cdot |I| = 1$. Let \mathcal{T} denote the set of all tiles.

We think of ω as a frequency interval and I as a spatial interval; our definition of a tile is a reflection of the uncertainty principle for the Fourier transform. We will plot frequency intervals in the vertical direction. Each dyadic interval ω is a union of two dyadic intervals of half the length of ω . We call them ω_{\pm} and view ω_{+} as above ω_{-} .

We take a fixed Schwartz function φ with frequency support in the interval $[-1/\nu, 1/\nu]$. For a tile $s = I_s \times \omega_s$, define

$$(3.2) \quad \varphi_s \stackrel{\text{def}}{=} \text{Mod}_{c(\omega_{s-})} \text{Tran}_{c(I_s)} \text{Dil}_{|I_s|}^2 \varphi.$$

Here, $c(J)$ is the center of the interval J , and ω_{s-} is the lower half of the interval ω_s . Thus, this function is localized to be supported in the time frequency plane close to the rectangle $I_s \times \omega_{s-}$.

There are companion functions which depend on different choices of certain measurable functions. These functions should be thought of as those choices of modulation and indices that will achieve, up to a constant multiple, the supremums in the oscillation function. To linearize the modulation, let

$N : \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function (a modulation parameter).

We define another function related to the rectangle $I_s \times \omega_{s+}$ which tells us when the linearized modulation parameter is at a certain frequency. Let

$$(3.3) \quad \phi_s(x) \stackrel{\text{def}}{=} \mathbf{1}_{\omega_{s+}}(N(x)) \varphi_s(x).$$

Now define a tile variant of the oscillation operator by

$$(3.4) \quad \text{Tile-osc}(f) \stackrel{\text{def}}{=} \left[\sum_{j=1}^{\infty} \sup_{k_j \leq l < l' < k_{j+1}} \left| \sum_{\substack{s \in \mathcal{T} \\ 2^l \leq |I_s| \leq 2^{l'}}} \langle f, \varphi_s \rangle \phi_s \right|^2 \right]^{1/2}.$$

Here, an increasing sequence of integers $\{k_j : j \geq 1\}$ are specified in advance. We make the definition for clarity's sake, as we will not explicitly work with it. Rather we prefer to fully linearize this maximal operator. This requires the additional choices of functions

$$(3.5) \quad \alpha_j : \mathbb{R} \longrightarrow \mathbb{R}, \quad \sum_{j=1}^{\infty} |\alpha_j(x)|^2 \leq 1, \quad \text{for all } x,$$

$$(3.6) \quad \ell_{j-}, \ell_{j+} : \mathbb{R} \longrightarrow \mathbb{Z}, \quad k_j \leq \ell_{j-} < \ell_{j+} < k_{j+1}.$$

And we set

$$(3.7) \quad F_{s,j} \stackrel{\text{def}}{=} \{x : 2^{\ell_{j-}(x)} \leq |I_s| < 2^{\ell_{j+}(x)}\},$$

$$(3.8) \quad f_{s,j}(x) \stackrel{\text{def}}{=} \mathbf{1}_{F_{s,j}}(x) \alpha_j(x) \phi_s(x).$$

The sequences of functions $\ell_{j\pm}$ are selecting the level at which the maximal difference occurs. The α_j are chosen to realize the ℓ^2 norm in the definition of oscillation. We make all of these choices in order to linearize the oscillation operator.

Our main proposition is

3.9. Proposition. *For all choices of $N(x)$ and increasing sequence of integers $\{k_j\}$, the operator Tile-osc extends to a bounded sub linear operator on L^p , $1 < p < \infty$. In particular, for sets $G, H \subset \mathbb{R}$ of finite measure, we have*

$$(3.10) \quad \sum_{\substack{s \in \mathcal{T} \\ j \in \mathbb{N}}} |\langle \mathbf{1}_G, \varphi_s \rangle \langle \mathbf{1}_H, f_{s,j} \rangle| \lesssim \min(|G|, |H|) (1 + |\log \frac{|G|}{|H|}|).$$

Note that the inequality above implies that

$$|\langle \text{Tile-osc}(\mathbf{1}_G), \mathbf{1}_H \rangle| \lesssim |G|^{1/p} |H|^{1-1/p}, \quad 1 < p < \infty.$$

That is, we have the restricted weak type inequality for all $1 < p < \infty$.¹ Hence, an interpolation argument will give us the estimate

$$(3.11) \quad \|\text{Tile-osc}(f)\|_p \lesssim \|f\|_p, \quad 1 < p < \infty.$$

The Deduction of Proposition 2.4 and Proposition 2.5. For $\xi \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, consider the operators

$$A_{\xi, \ell} f \stackrel{\text{def}}{=} \sum_{|I_s|=2^\ell} \mathbf{1}_{\xi \in \omega_{s+}} \langle f, \varphi_s \rangle \varphi_s.$$

The tile oscillation operator is built up from these operators. Observe that these operators enjoy the properties

$$(3.12) \quad A_{\xi, \ell} \text{Trans}_{n2^\ell} = \text{Trans}_{n2^\ell} A_{\xi, \ell}, \quad n \in \mathbb{Z},$$

$$(3.13) \quad A_{\xi, \ell} \text{Dil}_{2^{-\ell'}}^{(2)} = \text{Dil}_{2^{-\ell'}}^{(2)} A_{\xi 2^{-\ell'}, \ell + \ell'}, \quad \ell' \in \mathbb{Z},$$

$$(3.14) \quad A_{\xi, \ell} \text{Mod}_{-\theta} = \text{Mod}_{-\theta} A_{\xi + \theta, \ell}, \quad \theta \in \mathbb{R}.$$

Notice that these conditions tell us that the operators $A_{\xi, \ell}$ have a near translation invariance, a certain modulation invariance, and are related to each other through dilations. In addition, these operators are bounded on L^2 uniformly in ξ and ℓ , a fact well represented in the literature.

We will now define

$$B_{\xi, \ell} f \stackrel{\text{def}}{=} \lim_{\substack{K \rightarrow \infty \\ L \rightarrow \infty}} \frac{1}{4KL} \int_{-K}^K \int_{-L}^L \text{Mod}_{-\theta} \text{Trans}_{-y} A_{(\xi + \theta), \ell} (\text{Trans}_y \text{Mod}_{\theta} f) dy d\theta.$$

¹In fact, the estimate (3.10) gives a favorable upper bound on the behavior of the constant with respect to p , namely that they are no more than $\max(p, \frac{p}{p-1})$. See [12]

By periodicity of the integrand in y and θ , for all Schwartz functions f , the averages on the right hand side converge pointwise to $B_{\xi,l} f(x)$ as $K, L \rightarrow \infty$.

Let us make some observations about the operators $B_{\xi,l}$. First, (3.12) and periodicity of the integrand in y imply $B_{\xi,l}$ commutes with translations. Second, it is a bounded, positive, semidefinite operator, as is easy to see. Hence, it is given by convolution. Indeed, (3.14) implies that

$$B_{\xi,l} f = \text{Mod}_{\xi} \beta_l * (\text{Mod}_{-\xi} f).$$

for a function β_l that we turn to next. The equality (3.13) implies that $\beta_l = \text{Dil}_{2^l}^{(1)} \beta$ where β is given such that β_0 is a smooth Schwartz function satisfying the conditions (2.1)–(2.3), a routine exercise to verify.

Assuming Proposition 3.9, it follows that we can conclude Proposition 2.4 and Proposition 2.5 for nonzero functions $\psi = \beta_0$. Our proof is complete.

4. MAIN LEMMAS

To prove (3.10), we split the sum over $s \in \mathcal{T}$ into the sum over s such that $I_s \subset \{M \mathbf{1}_G > \lambda\}$ and the sum over s such that $I_s \not\subset \{M \mathbf{1}_G > \lambda\}$. The former sum can be taken care of by an argument of M. Lacey and C. Thiele [16] which also appears, slightly modified, in the paper of L. Grafakos, T. Tao, and E. Terwilleger [12]. Thus we restrict our attention to the tiles s where $I_s \not\subset \{M \mathbf{1}_G > \lambda\}$.

We begin with some concepts needed to phrase the proof. There is a natural partial order on tiles. We say that $s < s'$ iff $\omega_s \supset \omega_{s'}$ and $I_s \subset I_{s'}$. Note that the time variable of s is localized to that of s' , and the frequency variable of s is similarly localized, up to the variability allowed by the uncertainty principle. Note that two tiles are incomparable with respect to the ' $<$ ' partial order iff the tiles, as rectangles in the time frequency plane, do not intersect. A “maximal tile” will be one that is maximal with respect to this partial order.

Let \mathcal{S} denote an arbitrary set of tiles. We call a set of tiles $\mathbf{T} \subset \mathcal{S}$ a *tree* if there is a tile $I_{\mathbf{T}} \times \omega_{\mathbf{T}}$, called the *top of the tree*, such that for all $s \in \mathbf{T}$, $s < I_{\mathbf{T}} \times \omega_{\mathbf{T}}$. We note that the top is not uniquely defined. An important point is that a tree top specifies a location in time variable for the tiles in the tree, namely inside $I_{\mathbf{T}}$, and localizes the frequency variables, identifying $\omega_{\mathbf{T}}$ as a nominal origin.

We say that *the count of \mathcal{S} is at most A* iff $\mathcal{S} = \bigcup_{\mathbf{T} \subset \mathcal{S}} \mathbf{T}$, where each $\mathbf{T} \subset \mathcal{T}$ is a tree which is maximal with respect to inclusion and

$$\text{Count}(\mathcal{S}) \stackrel{\text{def}}{=} \sum_{\mathbf{T} \subset \mathcal{S}} |I_{\mathbf{T}}| \leq A.$$

Fix $\chi(x) = (1 + |x|)^{-\nu}$, where ν is, as before, a large constant whose exact value is unimportant to us. Define

$$(4.1) \quad \chi_I := \text{Trans}_{c(I)} \text{Dil}_{|I|}^{(1)} \chi,$$

$$(4.2) \quad \begin{aligned} \text{dense}(s) &:= \sup_{s < s'} \int_{N^{-1}(\omega_{s'}) \cap H} \chi_{I_{s'}} dx, \\ \text{dense}(\mathcal{S}) &:= \sup_{s \in \mathcal{S}} \text{dense}(s), \quad \mathcal{S} \subset \mathcal{T}. \end{aligned}$$

The first and most natural definition of a “density” of a tile, would be $|I_s|^{-1} |N^{-1}(\omega_{s+}) \cap I_s|$. However φ is supported on the whole real line, although it does decay faster than the inverse of any polynomial. We refer to this as a “Schwartz tails problem.” The definition of density as $\int_{N^{-1}(\omega_s)} \chi_{I_s} dx$, as it turns out, is still not adequate. That we should take the supremum over $s < s'$ only becomes evident in the proof of the “Tree Lemma” below.

The “Density Lemma” is

4.3. Lemma. *Any subset $\mathcal{S} \subset \mathcal{T}$ is a union of $\mathcal{S}_{\text{heavy}}$ and $\mathcal{S}_{\text{light}}$ for which*

$$\text{dense}(\mathcal{S}_{\text{light}}) < \frac{1}{2} \text{dense}(\mathcal{S}),$$

and the collection $\mathcal{S}_{\text{heavy}}$ satisfies

$$(4.4) \quad \text{Count}(\mathcal{S}_{\text{heavy}}) \lesssim \text{dense}(\mathcal{S})^{-1} |H|.$$

What is significant is that this relatively simple lemma admits a non-trivial variant intimately linked to the tree structure and orthogonality. We should refine the notion of a tree. Call a tree \mathbf{T} with top $I_{\mathbf{T}} \times \omega_{\mathbf{T}}$ a \pm tree iff for each $s \in \mathbf{T}$, aside from the top, $I_{\mathbf{T}} \times \omega_{\mathbf{T}} \cap I_s \times \omega_{s\pm}$ is not empty. Any tree is a union of a +tree and a -tree. If \mathbf{T} is a +tree, observe that the rectangles $\{I_s \times \omega_{s-} : s \in \mathbf{T}\}$ are disjoint. We see that

$$\sum_{s \in \mathbf{T}} |\langle f, \varphi_s \rangle|^2 \lesssim \|f\|_2^2.$$

This motivates the definition

$$(4.5) \quad \text{size}(\mathcal{S}) := \sup \{ |I_{\mathbf{T}}|^{-1/2} \sum_{s \in \mathbf{T}} |\langle f, \varphi_s \rangle|^2 : \mathbf{T} \subset \mathcal{S}, \mathbf{T} \text{ is a +tree} \}.$$

The “Size Lemma” is

4.6. Lemma. *Assume that $f = \mathbf{1}_G$. Any subset $\mathcal{S} \subset \mathcal{T}$ is a union of \mathcal{S}_{big} and $\mathcal{S}_{\text{small}}$ for which*

$$\text{size}(\mathcal{S}_{\text{small}}) < \frac{1}{2} \text{size}(\mathcal{S}),$$

and the collection \mathcal{S}_{big} satisfies

$$(4.7) \quad \text{Count}(\mathcal{S}_{\text{big}}) \lesssim \text{size}(\mathcal{S})^{-2} |G|.$$

Concerning the quantity size, we need an additional piece of information about it. Recall that M is the Hardy Littlewood maximal function.

4.8. Lemma. *Let $0 < \lambda < 1$, and suppose that \mathcal{S} is the set of tiles with*

$$I_s \not\subset \{M \mathbf{1}_G > \lambda\}, \quad s \in \mathcal{S}.$$

Then it is the case that $\text{size}(\mathcal{S}) \lesssim \lambda$.

This fact, a delicate consequence of the Calderon–Zygmund decomposition, will not be proved in this paper. It, like the Size Lemma and the Density Lemma, is already well represented in the literature. See, for example, [12]. For proofs of the Density and Size Lemmas, we refer the reader to [17]. The survey [15] is also suggested.

For a set of tiles \mathcal{S} , set

$$\text{Sum}(\mathcal{S}) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{N}} \sum_{s \in \mathcal{S}} |\langle \mathbf{1}_G, \varphi_s \rangle \langle \mathbf{1}_H, f_{s,j} \rangle|$$

Our final lemma relates trees, density and size. It is the “Tree Lemma.”

4.9. Lemma. *For any tree \mathbf{T}*

$$(4.10) \quad \text{Sum}(\mathbf{T}) \lesssim \text{size}(\mathbf{T}) \text{dense}(\mathbf{T}) |I_{\mathbf{T}}|.$$

Of course for any set of tiles \mathcal{S} , we would then have

$$\text{Sum}(\mathcal{S}) \lesssim \sum_{\mathbf{T} \subset \mathcal{S}} \text{size}(\mathbf{T}) \text{dense}(\mathbf{T}) |I_{\mathbf{T}}|.$$

Thus, we should inductively apply Lemma 4.3 and Lemma 4.6 so that the ‘Count’ estimates are essentially equal. The formal proof of Proposition 3.9, which is much as it appears in Lacey and Thiele [17] with the adaptation to a restricted weak type inequality as seen in [12], is left as an exercise for the reader.

5. PROOF OF LEMMA 4.9

The tree lemma, with its adaptation to the setting of oscillation, is the primary new step in this paper.

We begin with some remarks about oscillation operators, and a particular form of the same that we shall use at a critical point of this proof. Let ζ be a smooth function with Fourier transform supported in $[-1 - \varepsilon, 1 + \varepsilon]$ for a fixed, small, positive ε and equal to 1 on $[-1, 1]$. Set

$$\text{Osc}(f)^2 \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \sup_{2^{kj} \leq |I| \leq |I'| \leq 2^{k_{j+1}}} |\text{Dil}_{|I|}^{(1)} \zeta * f - \text{Dil}_{|I'|}^{(1)} \zeta * f|^2.$$

It is known that this is bounded on L^2 , and in this situation we will give an elementary proof of this fact below.

We shall have recourse to not only this bound, but a particular refinement. Let \mathcal{J} be a partition of \mathbb{R} into dyadic intervals. To each $J \in \mathcal{J}$, associate a subset $E(J) \subset J$ with $|E(J)| \leq \delta|J|$, where $0 < \delta < 1$ is fixed. Consider

$$(5.1) \quad \text{Osc}_\delta(f)^2 \stackrel{\text{def}}{=} \sum_{J \in \mathcal{J}} \mathbf{1}_{E(J)} \sum_{j=1}^{\infty} \sup_{\substack{J \subset I \subset I' \\ 2^{k_j} \leq |I| \leq |I'| \leq 2^{k_{j+1}}}} |\mathbf{1}_I \langle \zeta_I, f \rangle - \mathbf{1}_{I'} \langle \zeta_{I'}, f \rangle|^2$$

We estimate the norm of this operator.

5.2. Lemma. *We have the estimate*

$$(5.3) \quad \|\text{Osc}_\delta(f)\|_2 \lesssim \sqrt{\delta} \|f\|_2$$

for all $f \in L^2$.

Proof. Let us begin with a proof that $\|\text{Osc}\|_{2 \rightarrow 2} \lesssim 1$. That is, we do not have the additional information about the partition \mathcal{J} , and sets $E(J)$ for $J \in \mathcal{J}$. For a sequence of increasing integers k_j and function $f \in L^2$, set

$$\widehat{f}_j = \mathbf{1}_{2^{-k_{j+1}-1} \leq |\xi| \leq 2^{-k_j+1}} \widehat{f}$$

Then, we certainly have $\sum_{j \in \mathbb{N}} \|f_j\|_2^2 \leq 3\|f\|_2^2$. Moreover, due to our assumption about the function ζ ,

$$\sup_{2^{k_j} \leq |I| \leq |I'| \leq 2^{k_{j+1}}} |\text{Dil}_{|I|}^{(1)} \zeta * f| \leq M f_{j-1} + M f_j + M f_{j+1}$$

where M is the usual maximal function. Thus, by the boundedness of the maximal function on L^2 we have

$$\|\text{Osc}(f)\|_2^2 \leq 3 \sum_j \|M f_j\|_2^2 \lesssim \|f\|_2^2.$$

It is hardly surprising that the proof above appeals to the boundedness of the maximal function, as the estimate on the oscillation operator implies that for the maximal function. Likewise, our lemma implies a bound for a certain variant of the maximal function. As it turns out, we need this variant in the course of the proof.

Define

$$M_\delta f(x) \stackrel{\text{def}}{=} \sum_{J \in \mathcal{J}} \mathbf{1}_{E(J)}(x) \sup_{J \subset I} \langle |f|, \chi_I \rangle$$

where χ_I is defined as in (4.1). Then the estimate we claim is $\|M_\delta\|_2 \lesssim \sqrt{\delta}$.

Indeed, for any point $x \in E(J)$, we have the inequality

$$M_\delta f(x) \lesssim \inf_{y \in J} M f(y).$$

where M is the usual maximal function. Therefore, we can estimate

$$\begin{aligned} \|M_\delta f\|_2^2 &= \sum_{J \in \mathcal{T}} \int_{E(J)} M_\delta f(x)^2 dx \\ &\lesssim \sum_{J \in \mathcal{T}} |E(J)| \inf_{y \in J} M f(y)^2 \\ &\leq \delta \int M f(x)^2 dx. \end{aligned}$$

This proves our claim.

To conclude the proof, we can estimate

$$\begin{aligned} \int_{\cup_J E(J)} \text{Osc}_\delta(f)(x)^2 dx &\lesssim \sum_j \|M_\delta f_j\|_2^2 \\ &\lesssim \delta \sum_j \|f_j\|_2^2 \\ &\lesssim \delta \|f\|_2^2. \end{aligned}$$

Our proof is complete. □

We begin the main line of the argument. Let $\delta = \text{dense}(\mathbf{T})$, and $\sigma = \text{size}(\mathbf{T})$. By a modification of the functions $\alpha_j(x)$ by a choice of signs, we can assume the identity

$$\sum_{j \in \mathbb{N}} \sum_{s \in \mathbf{T}} |\langle \mathbf{1}_G, \varphi_s \rangle \langle f_{s,j}, \mathbf{1}_H \rangle| = \int_H \sum_{j \in \mathbb{N}} \sum_{s \in \mathbf{T}} \langle \mathbf{1}_G, \varphi_s \rangle f_{s,j}(x) dx.$$

As we have no particular control on the set H , we will need the following partition of the real line induced by the tree \mathbf{T} . Let \mathcal{J} be the partition of \mathbb{R} consisting of the maximal dyadic intervals J such that $3J$ does not contain any I_s for $s \in \mathbf{T}$. It is helpful to observe that for such J , if $|J| \leq |I_{\mathbf{T}}|$, then $J \subset 3I_{\mathbf{T}}$, and if $|J| \geq |I_{\mathbf{T}}|$, then $\text{dist}(J, I_{\mathbf{T}}) \gtrsim |J|$. The integral above is at most the sum of the two terms below.

$$(5.4) \quad \sum_{j \in \mathbb{N}} \sum_{J \in \mathcal{J}} \sum_{\substack{s \in \mathbf{T} \\ |I_s| \leq 2|J|}} |\langle \mathbf{1}_G, \varphi_s \rangle| \int_{J \cap H} |f_{s,j}(x)| dx$$

$$(5.5) \quad \sum_{j \in \mathbb{N}} \sum_{J \in \mathcal{J}} \int_{J \cap H} \left| \sum_{\substack{s \in \mathbf{T} \\ |I_s| > 2|J|}} \langle \mathbf{1}_G, \varphi_s \rangle f_{s,j}(x) \right| dx$$

Notice that for the second sum to be non-zero, we must have $J \subset 3I_{\mathbf{T}}$.

The first term (5.4) is controlled by an appeal to the ‘‘Schwartz tails.’’ Fix an integer $n \geq -1$, and only consider those $s \in \mathbf{T}$ for which $|I_s| = 2^{-n}|J|$. Recalling that $f_{s,j}(x) = \mathbf{1}_{F_{s,j}}(x)\alpha_j(x)\mathbf{1}_{\omega_{s+}}(N(x))\varphi_s(x)$, we see that

$$\begin{aligned} \sum_{j \in \mathbb{N}} \sum_{\substack{s \in \mathbf{T} \\ |I_s| = 2^{-n}|J|}} |\langle \mathbf{1}_G, \varphi_s \rangle| \int_{J \cap H} |f_{s,j}(x)| dx &\lesssim \sum_{\substack{s \in \mathbf{T} \\ |I_s| = 2^{-n}|J|}} \sigma \delta (|I_s|^{-1} \text{dist}(I_s, J))^{-10} |I_s| \\ &\lesssim \sigma \delta 2^{-n} \min(|J|, |J|(|I_{\mathbf{T}}|^{-1} \text{dist}(J, I_{\mathbf{T}}))^{-5}), \end{aligned}$$

Observe that for each s above, only one value of j contributes to the left hand sum. In addition, we have used the fact that there are only a bounded number of tiles s for which $|I_s|^{-1} \text{dist}(I_s, J)$ is essentially constant. In addition, for the case $|J| \leq |I_{\mathbf{T}}|$, we used that the distance from I_s to J is at least $\gtrsim |J|$. In the case $|J| > |I_{\mathbf{T}}|$, use $|I_s|^{-1} \text{dist}(I_s, J) \geq |I_{\mathbf{T}}|^{-1} \text{dist}(J, I_{\mathbf{T}})$. The estimate above can then be summed over $n \geq -1$ and $J \in \mathcal{J}$ to bound (5.4) by $\lesssim \sigma \delta |I_{\mathbf{T}}|$, as required.

Now we turn to the control of (5.5). The integral in this quantity is supported in the set

$$(5.6) \quad E(J) = J \cap \bigcup_{\substack{s \in \mathbf{T} \\ |I_s| > 2|J|}} (N^{-1}(\omega_{s+}) \cap H).$$

Then the critical observation is that $|E(J)| \lesssim \delta |J|$. To see this, let J' be the next larger dyadic interval that contains J . Then $3J'$ must contain some $I_{s'}$ for $s' \in \mathbf{T}$. Then there exists a tile s'' with $I_{s'} \subset I_{s''} \subset I_{\mathbf{T}}$ such that $|I_{s''}|$ is $2|J|$ or $4|J|$, and $\omega_{\mathbf{T}} \subset \omega_{s''} \subset \omega_{s'}$. Then, $s' < s''$, and by the definition of density,

$$\frac{|J \cap H \cap N^{-1}(\omega_{s''})|}{|I_{s''}|} \leq \int_{H \cap N^{-1}(\omega_{s''})} \chi_{I_{s''}} dx \leq \delta$$

But, for each s as in (5.6), we have $\omega_s \subset \omega_{s''}$, so that $E(J) \subset N^{-1}(\omega_{s''})$. Our claim follows.

Suppose that \mathbf{T} is a $-$ tree. This means that the tiles $\{I_s \times \omega_{s+} : s \in \mathbf{T}\}$ are disjoint and thus the functions $f_{s,j}$ are disjointly supported. In particular, the oscillation that arises from such functions is trivially bounded by their ℓ^∞ norm. Then the bound for (5.5) is no more than

$$\sum_{j \in \mathbb{N}} |E(J)| \left\| \sum_{\substack{s \in \mathbf{T} \\ |I_s| > 2|J|}} |\langle \mathbf{1}_G, \varphi_s \rangle f_{s,j}| \right\|_\infty \lesssim \delta \sigma |J|.$$

This is summed over $J \subset 3I_{\mathbf{T}}$ to get the desired bound.

Suppose that \mathbf{T} is a $+$ -tree. This is the interesting case. At this point, we will appeal to the norm bound for oscillation, (5.3), applied to the function

$$\Gamma \stackrel{\text{def}}{=} \text{Mod}_{-c(\omega_{\mathbf{T}})} \sum_{s \in \mathbf{T}} \langle \mathbf{1}_G, \varphi_s \rangle \varphi_s.$$

This is an assumption that can be assumed by an appropriate modulation of the fixed L^2 function f . In the definition of Γ , it is useful to us that we only use the “smooth” functions φ_s in the definition of this function. Note that $\|\Gamma\|_2 \lesssim \sigma \sqrt{|I_{\mathbf{T}}|}$, which is a consequence of the definition of size and the (near) orthogonality of the functions φ_s in the case of $+$ -tree.

The purpose of these next remarks is to relate the sums over a $+$ -tree to oscillation. Recall that the oscillation is defined relative to a sequence of integers k_j . For each J , consider $x \in J$ and integers ℓ such that $\max(2|J|, 2^{\ell j-(x)}) < 2^\ell < 2^{\ell j+(x)}$. We have

$$\sum_{\substack{s \in \mathbf{T} \\ |I_s|=2^\ell}} \langle \mathbf{1}_G, \varphi_s \rangle f_{s,j}(x) = \sum_{\substack{s \in \mathbf{T} \\ |I_s|=2^\ell}} \langle \mathbf{1}_G, \varphi_s \rangle \varphi_s(x) \alpha_j(x)$$

This is because all of the intervals ω_{s+} are nested and must contain $\omega_{\mathbf{T}}$, and if $N(x) \in \omega_{s+}$, then it must also be in every other $\omega_{s'+}$ that is the same size or larger. What is significant here is that on the right we have a particular scale of (a modulation of) the sum that defines Γ .

Furthermore, consider the functions

$$\Gamma_{j,J}(x) \stackrel{\text{def}}{=} \text{Mod}_{-c(\omega_{\mathbf{T}})} \sum_{\substack{s \in \mathbf{T} \\ \max(2|J|, 2^{\ell j-(x)}) \leq |I_s| \leq 2^{\ell j+(x)}}} \langle \mathbf{1}_G, \varphi_s \rangle \varphi_s.$$

In particular, we can choose ζ as in the definition of our oscillation operator (5.1) so that

$$\text{Dil}_{2^\ell}^{(1)} \zeta * \Gamma = \text{Mod}_{-c(\omega_{\mathbf{T}})} \sum_{\substack{s \in \mathbf{T} \\ |I_s| \geq 2^\ell}} \langle \mathbf{1}_G, \varphi_s \rangle \varphi_s.$$

Therefore, we have

$$\Gamma_{j,J} = \left[\text{Dil}_{\max(2|J|, 2^{\ell j-(x)})}^{(1)} \zeta - \text{Dil}_{2^{\ell j+(x)+1}}^{(1)} \zeta \right] * \Gamma.$$

We conclude that for $x \in E(J)$,

$$\begin{aligned} \sum_{j=1}^{\infty} \left| \sum_{\substack{s \in \mathbf{T} \\ 2|J| < |I_s|}} \langle \mathbf{1}_G, \varphi_s \rangle f_{s,j}(x) \right| &\leq \left(\sum_{j=1}^{\infty} |\Gamma_{j,J}(x)|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{\infty} |\alpha_j(x)|^2 \right)^{\frac{1}{2}} \\ &\lesssim \text{Osc}_\delta \Gamma(x), \end{aligned}$$

where we are using the oscillation operator defined in (5.1). We are able to use this operator here since $2|J| < |I_s|$ and $3J$ does not contain any I_s , which implies that $J \subset 3I_s$.

The conclusion of this proof is now at hand. By Lemma 5.2 we have

$$\begin{aligned}
\sum_{\substack{J \in \mathcal{J} \\ |J| \leq 3|I_{\mathbf{T}}|}} \int_{E(J)} \sum_{j=1}^{\infty} \left| \sum_{\substack{s \in \mathbf{T} \\ 2|J| < |I_s|}} \langle \mathbf{1}_G, \varphi_s \rangle f_{s,j}(x) \right| dx &\lesssim \int_{\bigcup_{|J| \leq 3|I_{\mathbf{T}}|} E(J)} |\text{Osc}_{\delta} \Gamma(x)| dx \\
&\lesssim \left| \bigcup_{|J| \leq 3|I_{\mathbf{T}}|} E(J) \right|^{1/2} \|\text{Osc}_{\delta} \Gamma\|_2 \\
&\lesssim \delta \sqrt{|I_{\mathbf{T}}|} \|\Gamma\|_2 \\
&\lesssim \sigma \delta |I_{\mathbf{T}}|.
\end{aligned}$$

6. CONCLUDING REMARKS

Let us pose a conjecture concerning the kernel $J_H(y) \stackrel{\text{def}}{=} \mathbf{1}_{[-1,1]} y^{-1}$, that is the Hilbert transform kernel with a sharp cut off.

6.1. Conjecture. *We have the inequality valid for all $n \geq 1$*

$$\|\text{Osc}_n(J_H; f)\|_p \lesssim \|f\|_p, \quad 1 < p < \infty.$$

In fact, the implied constant can be taken independent of n .

The proof as currently presented doesn't permit the deduction of this. Given the central role the Fourier transform plays in our proof, the technical difficulty we come to has a succinct description in terms of $\widehat{J_H}$. Namely, the variation of $\widehat{J_H}$ is infinite. But as the variation is only logarithmically infinite, one suspects that a proof of the conjecture above would have to revisit the proof of Carleson's theorem, with this example in mind.

6.2. Corollary. *For any measure preserving system (X, μ, T) and $f \in L^p(X, \mu)$ for $1 < p < \infty$, there is a set X_f of probability one for which for all $x \in X_f$*

$$\lim_{N \rightarrow \infty} \sum_{0 < |k| < N} \frac{e^{i\theta k}}{k} f(T^k x) \quad \text{exists for all } \theta.$$

The proof would begin by transferring the oscillation inequality in Theorem 1.4, valid on \mathbb{R} , to the integers \mathbb{Z} . This kind of transference can be done directly; it is also possible that the necessary result follows from known transference results such as Auscher and Carro [5]. Details are left to the reader.

Likewise, the method of proof that we employ throughout the paper could be adapted to shed light on more general singular integrals, as well as the original Wiener–Wintner Theorem. Indeed, an oscillation result could be proved for the latter theorem. We do not however pursue these lines here.

The Wiener–Wintner Theorem has a deep extension to the Return Time Theorem of Bourgain [6], see also the appendix to [7]. This Theorem, which we don’t recall in detail here, has certain extensions and variants that are currently only approachable via the phase plane methods of the type used in this paper. The Return Time is however a more sophisticated result, and the phase plane methods required are correspondingly more difficult. These issues will be explored in forthcoming papers of C. Demeter, M. Lacey, T. Tao, and C. Thiele.

REFERENCES

- [1] Idris Assani, *Caractérisation spectrale des systèmes dynamiques du type Wiener-Wintner*, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), 321–324. MR1821469 (2002c:37006) (French) ↑[3](#)
- [2] ———, *Wiener Wintner ergodic theorems*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003, ISBN 981-02-4439-8. MR1995517 (2004g:37007) ↑[3](#)
- [3] Idris Assani and Karl Petersen, *The helical transform as a connection between ergodic theory and harmonic analysis*, Trans. Amer. Math. Soc. **331** (1992), 131–142. MR 92g:28025 ↑
- [4] Idris Assani, Karl Petersen, and Homer White, *Some connections between ergodic theory and harmonic analysis*, Almost Everywhere Convergence, II (Evanston, IL, 1989), Academic Press, Boston, MA, 1991, pp. 17–40. MR 92m:28021 ↑
- [5] P. Auscher and M. J. Carro, *On relations between operators on \mathbf{R}^N , \mathbf{T}^N and \mathbf{Z}^N* , Studia Math. **101** (1992), 165–182. MR 94b:42007 ↑[16](#)
- [6] Jean Bourgain, *Temps de retour pour les systèmes dynamiques*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), 483–485. MR939436 (89e:28025) (French, with English summary) ↑[17](#)
- [7] ———, *Pointwise ergodic theorems for arithmetic sets*, Inst. Hautes Études Sci. Publ. Math. (1989), 5–45, With an appendix by the author, Harry Furstenberg, Yitzhak Katznelson and Donald S. Ornstein. MR1019960 (90k:28030) ↑[2](#), [3](#), [17](#)
- [8] A.-P. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. U.S.A. **59** (1968), 349–353. MR 37 #2939 ↑[2](#)
- [9] James Campbell and Karl Petersen, *The spectral measure and Hilbert transform of a measure-preserving transformation*, Trans. Amer. Math. Soc. **313** (1989), 121–129. MR 89i:28008 ↑[3](#)
- [10] James T. Campbell, Roger L. Jones, Karin Reinhold, and Máté Wierdl, *Oscillation and variation for the Hilbert transform*, Duke Math. J. **105** (2000), 59–83. MR1788042 (2001h:42021) ↑[2](#), [3](#)
- [11] Lennart Carleson, *On convergence and growth of partial sums of Fourier series*, Acta Math. **116** (1966), 135–157. 33 #7774 ↑[2](#)
- [12] Loukas Grafakos, Terrance Tao, and Erin Terwilleger, *L^p bounds for a maximal dyadic sum operator*, Math. Zeit. **246** (2004), <http://front.math.ucdavis.edu/math.CA/0212164>, 321–337. ↑[3](#), [4](#), [8](#), [9](#), [11](#)

- [13] Richard A. Hunt, *On the convergence of Fourier series*, Orthogonal Expansions and Their Continuous Analogues (Proc. Conf., Edwardsville, Ill., 1967), Southern Illinois Univ. Press, Carbondale, Ill., 1968, pp. 235–255. MR 38 #6296 ↑[2](#)
- [14] Roger L. Jones, Robert Kaufman, Joseph M. Rosenblatt, and Máté Wierdl, *Oscillation in ergodic theory*, Ergodic Theory Dynam. Systems **18** (1998), 889–935. MR1645330 (2000b:28019) ↑[2](#), [3](#)
- [15] Michael T. Lacey, *Carleson’s theorem: proof, complements, variations*, Publ. Mat. **48** (2004), <http://front.math.ucdavis.edu/math.CA/0307008>, 251–307. MR2091007 (2005e:42020) ↑[2](#), [11](#)
- [16] Michael T. Lacey and Christoph M. Thiele, *On Calderón’s conjecture*, Ann. of Math **149** (1999), <http://front.math.ucdavis.edu/math.CA/9903203>, 475–496. ↑[9](#)
- [17] ———, *A proof of boundedness of the Carleson operator*, Math. Res. Lett. **7** (2000), 361–370. MR 2001m:42009 ↑[3](#), [4](#), [11](#)
- [18] Attila Máté, *Convergence of Fourier series of square integrable functions*, Mat. Lapok **18** (1967), 195–242. MR 39 #701 (Hungarian, with English summary) ↑[3](#)
- [19] Malabika Pramanik and Erin Terwilleger, *A weak L^2 estimate for a maximal dyadic sum operator on \mathbb{R}^n* , Illinois J. Math. **47** (2003), 775–813. MR2007237 (2004g:42027) ↑[4](#)
- [20] Camil Muscalu, Terence Tao, and Christoph Thiele, *Multi-linear operators given by singular multipliers*, J. Amer. Math. Soc. **15** (2002), <http://front.math.ucdavis.edu/math.CA/9910039>, 469–496 (electronic). MR 2003b:42017 ↑[3](#)
- [21] Elena Prestini and Per Sjölin, *A Littlewood-Paley inequality for the Carleson operator*, J. Fourier Anal. Appl. **6** (2000), 457–466. MR1781088 (2001m:42017) ↑[4](#)
- [22] Norbert Wiener and Aurel Wintner, *On the ergodic dynamics of almost periodic systems*, Amer. J. Math. **63** (1941), 794–824. MR 4,15a ↑[2](#)

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